Equilibrium of concurrent forces

When a structure or a machine is at rest, it is said to be in static equilibrium. In general, the state of equilibrium can be defined as a state of rest or balance under the action of forces which counteract each other.

This chapter is concerned with conditions of equilibrium of concurrent forces. A system of forces is called concurrent when the lines of action of all forces intersect at a common point, the point of concurrency.

4.1 Conditions of equilibrium

For a system of concurrent forces to be in equilibrium, the resultant force, that is the result of summation of all forces acting through that point, must be equal to zero. In other words, all forces balance each other in such a way that there is no resultant push or pull acting on the body at the point of application of the forces.

To prove that a given system of forces is in fact in equilibrium, we must demonstrate graphically or mathematically that the forces add up to zero. Naturally, since forces are vector quantities, the addition must be vectorial addition as explained in Chapter 3. Consider the following example.

Example 4.1

A joint of a pin-jointed truss is as shown in Figure 4.1(a) and has internal and external forces as shown. Prove that the joint is in equilibrium.

![Figure 4.1](image)

**Solution**

In order to prove the equilibrium of this system of concurrent forces graphically, it is necessary to construct the polygon of forces, using all of the applied forces, external ($F_i$) as well as internal forces ($F_2$, $F_3$ and $F_4$), and to show that the starting point of the construction coincides with the end point, i.e. the polygon must close, leaving no room for a gap representing a resultant force (remember: the resultant must be zero). See Figure 4.1(b).

Mathematically the summation of forces is signified by $\Sigma F$, meaning that the resultant force is the sum of all the applied forces. For a system in equilibrium, the resultant is equal to zero, or

\[ \Sigma F = 0 \]

This is the equation of forces in equilibrium, which is a very useful equation. However, this equation is often interpreted in terms of the perpendicular components of the resultant force. If the resultant is equal to zero, its components must also be equal to zero, i.e. no force—no components. It follows that, for a given system of forces, the sum of all components in any direction must be zero. This is usually expressed in terms of mutually perpendicular directions $x$ and $y$, often horizontal and vertical.

\[ \Sigma F_x = 0 \quad \text{and} \quad \Sigma F_y = 0 \]

Now let us consider a mathematical solution to the previous example.

**Alternative solution**

To prove the equilibrium of forces mathematically, it is necessary to demonstrate that both vertical and horizontal components add up to zero. We can use a table similar to Table 3.1 to record all the force components, including their positive and negative signs. The solution will appear as in Table 4.1.

The sum of the horizontal, or $x$ direction, components and the sum of the vertical, or $y$ direction, components have been shown to be zero, i.e. the two equations $\Sigma F_x = 0$ and $\Sigma F_y = 0$ have each been satisfied. The conclusion is that the system of forces is in equilibrium, as we would expect in a structure such as a roof truss.

<table>
<thead>
<tr>
<th>Force</th>
<th>Magnitude</th>
<th>$X$ component</th>
<th>$Y$ component</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_1$</td>
<td>3.0</td>
<td>0</td>
<td>-3.0</td>
</tr>
<tr>
<td>$F_2$</td>
<td>3.5</td>
<td>-3.03</td>
<td>-1.75</td>
</tr>
<tr>
<td>$F_3$</td>
<td>3.0</td>
<td>-2.60</td>
<td>1.50</td>
</tr>
<tr>
<td>$F_4$</td>
<td>6.5</td>
<td>5.63</td>
<td>3.25</td>
</tr>
</tbody>
</table>

| \[ \Sigma F_x = 0 \quad \Sigma F_y = 0 \] |
4.2 Free-body diagrams

When problems involving the equilibrium of bodies under the action of force systems are being solved, a method of setting out the essential details in the form of a diagram is necessary.

We often start with a semi-pictorial sketch, or space diagram showing the physical conditions of the problem, i.e. the layout of its mechanical or structural components such as pulleys, supports, cables, rollers, etc.

The next step is to isolate the essential facts about the forces involved and to draw a free-body diagram. Such a diagram usually shows the point of concurrency acted upon by all the forces, indicating the magnitudes and directions of the forces.

A force polygon is then constructed, based on the information contained in the free-body diagram.

Example 4.2

Consider the equilibrium of forces and hence determine the force in each cable, when two cranes are supporting a 5 tonne mass as shown in Figure 4.2(a).

![Fig. 4.2](a) Space diagram (b) Free-body diagram (c) Force triangle

Solution

The weight of the load is:

$$F_w = m \cdot g = 5000 \text{ kg} \times 9.81 \frac{\text{N}}{\text{kg}} = 49,050 \text{ N} = 49.05 \text{ kN}$$

The free-body diagram, Figure 4.2(b), is drawn to represent all forces acting on the point of concurrency. Note that each force is represented as a vector, showing its magnitude and direction in relation to the point.

Knowing one force, i.e. the weight, and the direction of the other two forces enables us to construct the force triangle. (Note that a triangle is just a special case of force polygon where the number of forces involved is three.)

If the triangle of forces is drawn to scale, the answers can easily be scaled off. Alternatively, the sine rule can be used to calculate the unknown forces as side-lengths of the force triangle.

$$F_1 = 49.05 \times \frac{\sin 60^\circ}{\sin 75^\circ} = 43.98 \text{ kN}$$

$$F_2 = 49.05 \times \frac{\sin 45^\circ}{\sin 75^\circ} = 35.91 \text{ kN}$$

Example 4.3

An elastic member $ABC$ is stretched as shown in Figure 4.3(a) by three forces $F_1$, $F_2$ and $F_3$.

Determine the forces in $AB$ and $BC$.

Solution

In this example more than three forces are involved. Therefore, a polygon of forces, rather than a triangle would have to be constructed, as in Figure 4.3(b).

The procedure is to construct as much of the polygon as possible using all known forces and then to complete it by drawing two lines, parallel to the unknown forces, through the first and last points.

The answers are scaled off the force polygon and are equal to $F_{AB} = 148 \text{ N}$ and $F_{BC} = 167 \text{ N}$.*

![Fig. 4.3](a) (b)

4.3 The equilibrant force

If a system of concurrent forces is not balanced, i.e. not in equilibrium, it is possible to determine the additional force required to produce equilibrium. Such a force is called the equilibrant. In other words, the equilibrant of a system of concurrent forces is that force which when added to the system produces equilibrium.

In the event of a balanced system of forces, the equilibrant force would be equal to zero.

Example 4.4

Determine the equilibrant force for the system of forces shown in Figure 4.4(a).

* An alternative mathematical solution is also possible, but is rather complex, involving two simultaneous equations in terms of unknown forces. It is not recommended at this level.
PART TWO Statics

Fig. 4.4

**Solution**

To solve this problem graphically, we must recall that for a system of forces in equilibrium the force polygon must close. If we attempt to construct the force polygon, using only the given forces $F_1$, $F_2$ and $F_3$, arranging the forces in head-to-tail order, we find that the first and last points do not coincide. In order to close the gap an additional line is required, as shown in Figure 4.4(b). This line will close the polygon and represent the required equilibrant force, both in magnitude and direction.

The system of forces in equilibrium is shown in Figure 4.4(c).

It is important to remember that the closed polygon of forces has all its forces, including the equilibrant force, follow the head-to-tail order. This enables us to determine the correct direction of the equilibrant force, e.g. in this example the equilibrant is a vertical downward force.

It should also be understood that the equilibrant and the resultant forces are always equal in magnitude and opposite in direction. (Compare Examples 3.6 and 4.3)

**Alternative solution**

The equilibrant force can also be found mathematically by means of addition of rectangular components, including those for the unknown equilibrant force.

The solution is best set out in tabular form as in Table 4.2.

<table>
<thead>
<tr>
<th>Force</th>
<th>Magnitude</th>
<th>$X$ component</th>
<th>$Y$ component</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_1$</td>
<td>258.0 N</td>
<td>-223.4 N</td>
<td>129.0 N</td>
</tr>
<tr>
<td>$F_2$</td>
<td>110.3 N</td>
<td>78.0 N</td>
<td>78.0 N</td>
</tr>
<tr>
<td>$F_3$</td>
<td>145.4 N</td>
<td>145.4 N</td>
<td>0</td>
</tr>
<tr>
<td>Equilibrant</td>
<td>$F_x$</td>
<td>$F_y$</td>
<td>$F_z$</td>
</tr>
</tbody>
</table>

For equilibrium: $\sum F_x = 0 - 223.4 + 78.0 + 145.4 + F_x = 0$

Hence $F_x = 0$

Also $\sum F_y = 0 + 129.0 + 78.0 + 0 + F_y = 0$

Hence $F_y = -207$ N

The magnitude of the equilibrant force is therefore

$$F_z = \sqrt{F_x^2 + F_y^2} = \sqrt{0 + 207^2} = 207$$ N

Consideration of the directions of components $F_x$ and $F_y$ indicates that the equilibrant is acting vertically downwards.

**Problems**

4.1 A light fitting having a mass of 1.5 kg is hanging from the ceiling on wire $AB$ and tied to the wall by string $BC$ (Fig. 4.5). Determine the forces in $AB$ and $BC$.

**Fig. 4.5**

4.2 A street light of mass 15 kg is supported at mid-point between two poles by a cable $ABC$ (Fig. 4.6). If the length of the cable $ABC$ is 20 m and deflection $BD$ at mid-point is 0.2 m, determine the force in the cable.
4.3 The jib-crane in Figure 4.7 carries a load of 1.4 tonne. Determine the forces in the jib and the tie.

4.4 Determine the forces in \( AB, BC, CD \) and \( BE \) (Fig. 4.8) if the mass \( m = 500 \text{ kg} \).

4.5 Determine the force acting through the axis of the pulley shown (Fig. 4.9). Does the magnitude or direction of the force depend on pulley diameter?

4.6 Determine the mass lifted and the forces in members \( AB \) and \( BC \) (Fig. 4.10) when the tension in the cable is 20 kN.

4.7 Figure 4.11 represents joints in a simple pin-jointed roof truss. Using the information available for each joint, determine the unknown forces by constructing a separate polygon of forces for each joint.
4.4 Support reactions

Before proceeding to the next section, it is advisable to discuss the types of contact surfaces or supports in relation to the kind of reaction force that may exist between a body, such as a beam or a truss, and its supporting surface.

In this chapter we need to consider two categories of supports: those at which the reaction force is always normal to the supporting surface, and those which may support a force at any angle to the supporting surface.

The first category includes a smooth, frictionless surface contact between a body and a supporting surface, such that the body can slide along the surface at the point of contact without any resistance. A support on rollers, such as one can see under some long bridge sections, has a similar force action, i.e. it provides normal** reaction only.

The second category includes a rough surface contact between a body and a supporting surface in which friction prevents relative sliding motion, producing a reaction force in any direction. A fixed hinge or bearing has a similar force action capable of supporting a force at any angle to the supporting surface.

The rule to remember is that, at a smooth surface or at a roller support, the reacting force is always perpendicular to the supporting surface.

In addition to the above-mentioned reaction forces, it is helpful to recognize that forces in links pivoted at each end and in cables connecting two points of a structure always act along the axes of such members (see Table 4.3).

Table 4.3 Classification of supports, connections and contact surfaces

1. Normal reaction only
   (a) smooth contact surface
   (b) ball or roller support

2. Force along axis of member
   (a) cable
   (b) link

3. Reaction at any angle
   (a) rough contact surface
   (b) pin or hinge

* Friction is discussed in Chapter 7.
** The normal to a line or surface is a direction perpendicular to it, e.g. normal force is perpendicular to the supporting surface.